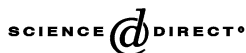




ELSEVIER

Available online at www.sciencedirect.com



Journal of Number Theory 107 (2004) 105–134

JOURNAL OF
**Number
Theory**

<http://www.elsevier.com/locate/jnt>

A two-dimensional Minkowski $\?(x)$ function

Olga R. Beaver and Thomas Garrity*

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267, USA

Received 7 April 2003

Communicated by A. Granville

Abstract

A function from a triangle to itself is defined that has both interesting number theoretic and analytic properties. This function is shown to be a natural generalization of the classical Minkowski $\?(x)$ function. It is shown there exists a natural class of pairs of cubic irrational numbers in the same cubic number field that are mapped to pairs of rational numbers, in analog to $\?(x)$ mapping quadratic irrationals on the unit interval to rational numbers on the unit interval. It is also shown that this new function satisfies an analog to the fact that $\?(x)$, while increasing and continuous, has derivative zero almost everywhere.

© 2004 Elsevier Inc. All rights reserved.

1. Introduction

Any real number with an eventually periodic continued fraction expansion must be a quadratic irrational. This property linking periodicity of a number's continued fraction expansion with its being quadratic led Minkowski to define his remarkable question-mark function

$$\? : [0, 1] \rightarrow [0, 1].$$

(See [27, Vol. II, p. 50]; see also [15, article 196, p. 754], which appears to be essentially a translation of all of Minkowski's number theory papers.) The question-mark function is increasing, continuous, maps each rational number $\frac{p}{q}$ to a pure dyadic number of the form $\frac{k}{2^n}$, maps each quadratic irrational to a rational number, and has the property that the inverse image of the rational numbers is exactly the set of quadratic irrationals. In order to understand the number theoretic properties of

*Corresponding author. Fax: +1-413-597-4061.

E-mail addresses: obeaver@williams.edu (O.R. Beaver), tgarrity@williams.edu (T. Garrity).

quadratic irrationals, it is natural to look at the function theoretic properties of $?(x)$. In particular, the question-mark function is not only continuous and monotonically increasing but has derivative zero almost everywhere. As such, it is a naturally occurring example of a singular function. Moreover, it is, in fact, the diophantine properties of continued fractions that lead to its derivative being zero a.e. Thus, the analytic property of $?(x)$ being both increasing and having derivative zero almost everywhere is actually number theoretic in origin.

In this paper, we will construct a function similar to Minkowski's question-mark function, and will use that function in order to understand the properties of cubic irrationals.

Denjoy [4,5] and independently Salem [36] were the first to realize that $?(x)$ is singular, although earlier, Ryde [34] proved in essence that $?(x)$ was singular. However, Ryde showed that $?(x)$ was singular without realizing its connection with Minkowski's function (see also [35]). Recent work on $?(x)$ is the work of Kinney [16], Girgensohn [9], Ramharter [33], of Tichy and Uitz [41], and of Viader et al. [42], Paradis et al. [31]. (In fact, the idea for the inequality that we prove in Section 6.1 and use in 6.2 was inspired by the work of Viader et al. [42].)

A natural question to ask is: do cubic irrationals and other higher order algebraic numbers lend themselves to similar analysis? An even more basic question to ask is how to generalize the relation between periodicity for continued fractions and quadratic irrationals to cubics. In 1848, in a letter to Jacobi, Hermite [14] asked for such a generalization. Specifically, the Hermite problem is:

Find methods for expressing real numbers as sequences of positive integers so that the sequence is eventually periodic precisely when the initial number is a cubic irrational.

Over the years there has been much work in trying to solve the Hermite problem. For an overview, see Schweiger's *Multidimensional Continued Fractions* [40]. For work up to 1980, see Brentjes' overview in [3]. Other work is in [1,2,6–8,11–13,17,19–25,28,29,32,37–39].

On the other hand, there has been little attempt to approach the Hermite problem by generalizing the Minkowski $?(x)$. The only such attempt that we have found is in the thesis of Kollros [18]. Kollros generalizes $?(x)$ to a map from the unit square to itself. However, while he sets up various methods for associating points in the unit square with sequences of integers, he does not concern himself with the function-theoretic properties of this function. It does not appear that Kollros has solved the Hermite problem. In particular, he was not interested in the differentiability properties of his analogue to $?(x)$.

In this paper we develop a different, more natural, analog to $?(x)$. In Section 2, a review of the Minkowski question-mark function is given. In Section 3, we construct a map from a two-dimensional simplex (a triangle) to itself, as an analog to the map of $?(x)$ from a one-dimensional simplex to itself. The map will be determined by partitioning the triangle, first via a "Farey" partition, and then by a barycentric (triadic) partitioning, which we will frequently call the "bary" partitioning. We define a function $\delta(x, y)$ from the Farey triangle to the barycentric triangle. In

Section 4, we see that the Farey iteration can be viewed as a multidimensional continued fraction. We show that periodicity of the Farey iteration corresponds (usually) to a class of cubic irrationals. In Section 5 we show, by contrast, that periodicity for the barycentric iterations corresponds to a class of rational points. This results in that our function will map a natural class of cubic points to a natural class of rational points. Finally, in Section 6, we prove an analog of singularity by showing that, a.e., the area of image triangles in the barycentric partitioning approaches zero far more quickly than the area of the domain triangles in the Farey partitioning.

We note that similar, but not exactly the same, type of Farey partitioning, or Farey nets, to solve the Hermite problem has been considered by both Monkemeyer [30] and more recently by Grabiner [10]. Both papers are quite interesting; neither use Farey nets in the way that we do and, more importantly for this paper, neither attempt to generalize the Minkowski $\varphi(x)$ function. In actuality, this analytic approach would not have been a natural succession in either of these papers, as Monkemeyer's and Grabiner's goals were not function theoretic.

2. A review of the Minkowski question-mark function

All of the discussion in this section is well-known. We include it here for sake of completeness.

Recall that given two rational numbers $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$, each in lowest terms, the *Farey sum*, $\hat{+}$, of the numbers is

$$\frac{p_1}{q_1} \hat{+} \frac{p_2}{q_2} = \frac{p_1 + p_2}{q_1 + q_2}.$$

The φ function is then defined as follows. Suppose we know the value of $\varphi(\frac{p_1}{q_1})$ and $\varphi(\frac{p_2}{q_2})$. We then set

$$\varphi\left(\frac{p_1 + p_2}{q_1 + q_2}\right) = \frac{\varphi(\frac{p_1}{q_1}) + \varphi(\frac{p_2}{q_2})}{2}.$$

Specifying the initial values

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(1) = 1,$$

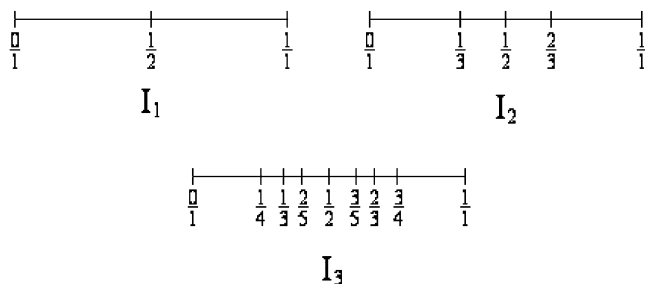
we now know the values of $\varphi(x)$ for any rational number x .

By continuity arguments we can determine the values of $\varphi(x)$ for any real number x in the unit interval. Since we will be generalizing this continuity argument in the next section, we discuss this now in some detail.

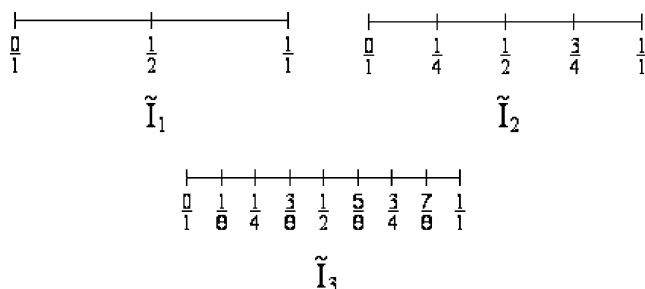
We produce two sequences of partitions, \mathcal{I}_k and $\tilde{\mathcal{I}}_k$, of the unit interval. For each $k \geq 0$, each partition will split the unit interval into 2^k subintervals. Both start with just the unit interval itself:

$$\mathcal{I}_0 = \tilde{\mathcal{I}}_0 = [0, 1].$$

Note that $0 = \frac{0}{1}$ and $1 = \frac{1}{1}$. Now, given the partition \mathcal{J}_{k-1} , suppose that the endpoints of each of its 2^{k-1} open subintervals are rational numbers. Form the next partition \mathcal{J}_k by taking the Farey sum of the endpoints of the partition \mathcal{J}_{k-1} . Thus the endpoints of \mathcal{J}_k consist of the Farey fractions of order k .



The partition $\tilde{\mathcal{J}}_k$ is even simpler. It is just the partition given by the subintervals $[\frac{l-1}{2^k}, \frac{l}{2^k}]$.



Then the function $?(x)$ can be seen to map the endpoints of each \mathcal{J}_k to the corresponding endpoints of $\tilde{\mathcal{J}}_k$.

Now, as is shown, for example, in [36], $?(x)$ is singular and hence has derivative zero almost everywhere. Using the partitions defined above, we can recast the fact that $?(x)$ is a singular function into the language of lengths of intervals. Fix $\alpha \in [0, 1]$. For each k , let I_k and \tilde{I}_k be the subintervals of the respective partitions \mathcal{J}_k and $\tilde{\mathcal{J}}_k$ that contain the point α . Then, as shown in [36, p. 437].

Theorem 1. For almost all $\alpha \in [0, 1]$,

$$\liminf_{k \rightarrow \infty} \frac{\text{length of } \tilde{I}_k}{\text{length of } I_k} = 0.$$

It is this theorem that provides the most natural language for generalizing the failure of differentiability for our analog of the question-mark function.

The proof involves the idea that the Diophantine approximation properties of continued fractions make the above denominator approach zero more slowly than the numerator.

3. The Farey–Bary map: a generalization of the Minkowski question-mark function

Our goal is to define a map from a two-dimensional simplex (a triangle) to itself that generalizes the Minkowski question-mark function. This will involve two separate partitionings of the triangle. We would like to have periodicity in the domain correspond to cubic irrationals while periodicity in the range to imply rationality. Both of these goals will only be achieved in part, as we will show that periodicity will usually imply cubic irrationality in the domain case and will always imply rationality for the range. At the same time, we want our generalization to obey some sort of singularity property.

Although in a sense it would be most natural to denote our generalization by the symbol $?(x, y)$, we have found that it is both awkward to say and awkward to read. Thus we will denote our generalization by $\delta(x, y)$.

3.1. The Farey sum in the plane

We will often need to refer to a point in the plane of the form

$$v = \begin{pmatrix} p/r \\ q/r \end{pmatrix}.$$

Here, since the coordinates share the same denominator, we can associate to this point a unique vector in space, namely

$$\bar{v} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

Conversely, a vector $\bar{v} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ can be associated uniquely to the point

$$v = \begin{pmatrix} p/r \\ q/r \end{pmatrix}$$

in the plane. In what follows, we will usually refer to both the point and its corresponding vector as v .

Consider three points in the plane, each of whose entries are nonnegative integers, each $r_i \neq 0$, and such that each vector's entries share no common factors:

$$v_1 = \begin{pmatrix} p_1/r_1 \\ q_1/r_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} p_2/r_2 \\ q_2/r_2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} p_3/r_3 \\ q_3/r_3 \end{pmatrix}.$$

These points define a triangle in the plane and, as noted above, can also be represented as the vectors,

$$v_1 = \begin{pmatrix} p_1 \\ q_1 \\ r_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} p_2 \\ q_2 \\ r_2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} p_3 \\ q_3 \\ r_3 \end{pmatrix}.$$

Summing the three vectors, we get

$$v = v_1 + v_2 + v_3 = \begin{pmatrix} p_1 + p_2 + p_3 \\ q_1 + q_2 + q_3 \\ r_1 + r_2 + r_3 \end{pmatrix}.$$

This vector sum can be converted into a point v in the plane, where

$$v = \begin{pmatrix} \frac{p_1 + p_2 + p_3}{r_1 + r_2 + r_3} \\ \frac{q_1 + q_2 + q_3}{r_1 + r_2 + r_3} \end{pmatrix}.$$

This correspondence between points in the plane and a vector representation allows us to define the Farey sum.

Definition 2. Let

$$v_1 = \begin{pmatrix} p_1/r_1 \\ q_1/r_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} p_2/r_2 \\ q_2/r_2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} p_3/r_3 \\ q_3/r_3 \end{pmatrix},$$

where, for each i , the p_i , q_i and r_i share no common factor. The Farey sum, \hat{v} of the v_i is then

$$\hat{v} = v_1 + v_2 + v_3 = \begin{pmatrix} \frac{p_1 + p_2 + p_3}{r_1 + r_2 + r_3} \\ \frac{q_1 + q_2 + q_3}{r_1 + r_2 + r_3} \end{pmatrix}.$$

Note that the point \hat{v} is inside the triangle determined by the vertices v_1, v_2 and v_3 and that \hat{v} corresponds to the vector $\begin{pmatrix} p_1 + p_2 + p_3 \\ q_1 + q_2 + q_3 \\ r_1 + r_2 + r_3 \end{pmatrix}$.

3.2. Farey and barycentric partitions

In this section we will define two partitions of the triangle

$$\Delta = \{(x, y) : 1 \geq x \geq y \geq 0\}.$$

The first partition of Δ will yield the domain of our desired function δ , while the second partition will yield the range.

3.2.1. The Farey partition

We will define a sequence of partitions $\{\mathcal{P}_n\}$ such that each \mathcal{P}_n will consist of 3^n subtriangles of Δ and each \mathcal{P}_n will be a refinement of the previous \mathcal{P}_{n-1} .

Let \mathcal{P}_0 be the initial triangle Δ . The three vertices of Δ are

$$v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0/1 \\ 0/1 \end{pmatrix},$$

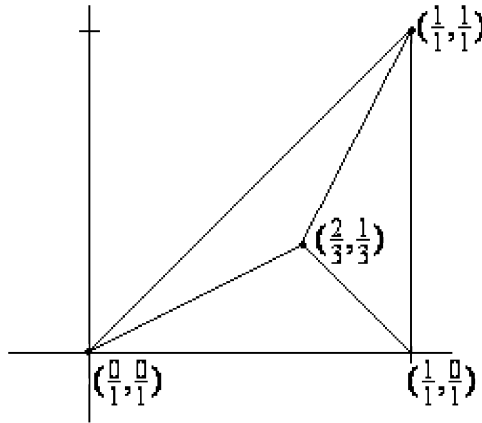
$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/1 \\ 0/1 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/1 \\ 1/1 \end{pmatrix}.$$

Taking the Farey sum of these vertices, we have

$$v_1 \hat{+} v_2 \hat{+} v_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

This Farey vector corresponds to the point $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$. In particular, the point $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$ is an interior point of the triangle Δ and, in a natural way, partitions Δ into three subtriangles. We will refer to the resulting interior point as the *Farey-center*.



This determines the partition \mathcal{P}_1 . We now proceed inductively. Suppose we have the partition \mathcal{P}_n , that determines 3^n triangles. We now partition each of these triangles into three subtriangles, as follows. Suppose one of the triangles in \mathcal{P}_n has vertices $\begin{pmatrix} p_1/r_1 \\ q_1/r_1 \end{pmatrix}$, $\begin{pmatrix} p_2/r_2 \\ q_2/r_2 \end{pmatrix}$ and $\begin{pmatrix} p_3/r_3 \\ q_3/r_3 \end{pmatrix}$. Computing the Farey sum of the three vertices of the triangle gives a point \hat{v} , the Farey-center, in the interior of the subtriangle. The Farey-center, \hat{v} , yields a partition of the subtriangle. Computing in this way the partition of each subtriangle of Δ determined by \mathcal{P}_n , gives us the desired next partition \mathcal{P}_{n+1} of Δ .

We denote this full partitioning of Δ by Δ_F and call it the *Farey partitioning*.

3.2.2. The Barycentric partition

Again we will define a sequence of partitions $\tilde{\mathcal{P}}_n$ of Δ such that each $\tilde{\mathcal{P}}_n$ will consist of 3^n triangles of Δ and each $\tilde{\mathcal{P}}_n$ will be a refinement of the previous $\tilde{\mathcal{P}}_{n-1}$.

As with the Farey partitioning, the zeroth level partition $\tilde{\mathcal{P}}_0$ is simply the initial triangle Δ .

To compute $\tilde{\mathcal{P}}_1$, we again start with the original three vertices of Δ and compute the barycenter of Δ , namely $\left(\frac{2}{3}\right)$. This point happens, in this case, to be the same point obtained by computing the Farey sum of the coordinates of the vertices of Δ , the Farey-center. This is just a coincidence.

Proceed inductively as follows. Assume we have a partition $\tilde{\mathcal{P}}_n$ of Δ into 3^n subtriangles. Further, assume at the n th stage that the coordinates of the vertices of any subtriangle can be expressed as rational numbers with 3^n in the denominator. Then, if a given subtriangle in $\tilde{\mathcal{P}}_n$ has vertices $\left(\frac{a_1}{3^n}, \frac{a_2}{3^n}, \frac{a_3}{3^n}\right)$, $\left(\frac{b_1}{3^n}, \frac{b_2}{3^n}, \frac{b_3}{3^n}\right)$, and $\left(\frac{c_1}{3^n}, \frac{c_2}{3^n}, \frac{c_3}{3^n}\right)$, we compute the Farey sum of the three vertices. This again gives the barycenter of the subtriangle, namely, $\left(\frac{a_1+a_2+a_3}{3^{n+1}}, \frac{b_1+b_2+b_3}{3^{n+1}}, \frac{c_1+c_2+c_3}{3^{n+1}}\right)$. Computing, in this way, the partition of each

subtriangle of $\tilde{\Delta}$ determined by $\tilde{\mathcal{P}}_n$ gives us the desired next partition $\tilde{\mathcal{P}}_{n+1}$ of Δ .

We call this full partitioning of Δ the *barycentric*, or *Bary*, partitioning and denote it by Δ_B .

3.3. The Farey–Bary map

We are now ready to define the extension of the Minkowski question-mark function to $\delta : \Delta_F \rightarrow \Delta_B$. We will proceed in stages. At first, we will define a function $\delta_n(x, y)$ from the vertices of the n th partition of Δ_F to the vertices of the n th partition of Δ_B and then extend linearly δ_n to all of Δ_F . Then we will use these functions $\{\delta_n\}$ to define our desired function $\delta(x, y)$ on Δ_F .

We first need to introduce some notation. Each of the partitions \mathcal{P}_n and $\tilde{\mathcal{P}}_n$ determines subtriangles of Δ_F and Δ_B , respectively. Let $\Delta_{n,F}$ and $\Delta_{n,B}$ denote Δ_F and Δ_B after the n th partitioning, respectively. The expression $\langle v_1(n), v_2(n), v_3(n) \rangle$ will denote a general subtriangle of $\Delta_{n,F}$ with vertices $v_1(n)$, $v_2(n)$, and $v_3(n)$. When we need to refer to the 3^n specific subtriangles, we will use $\langle v_1^s(n), v_2^s(n), v_3^s(n) \rangle$, where $s \in \{1, \dots, 3^n\}$. In a similar fashion, we will refer to the subtriangles of $\tilde{\mathcal{P}}_n$ by $\langle \tilde{v}_1(n), \tilde{v}_2(n), \tilde{v}_3(n) \rangle$, in the general case, and $\langle \tilde{v}_1^s(n), \tilde{v}_2^s(n), \tilde{v}_3^s(n) \rangle$ in the specific case.

Note that it happens to be the case that

$$\mathcal{P}_0 = \tilde{\mathcal{P}}_0, \quad \mathcal{P}_1 = \tilde{\mathcal{P}}_1 \quad \text{and} \quad \Delta_{1,F} = \Delta_{1,B}.$$

Definition 3. Define $\delta_0, \delta_1 : \Delta_F \rightarrow \Delta_B$ to be the identity maps on the vertices of the subtriangles determined by \mathcal{P}_0 and \mathcal{P}_1 . For any n , define δ_n to send any vertex in the

n th partition \mathcal{P}_n to the corresponding vertex in the partition $\tilde{\mathcal{P}}_n$. That is, define δ_n on any subtriangle $\langle v_1(n), v_2(n), v_3(n) \rangle$ of the partition \mathcal{P}_n by

$$\delta_n(v_i(n)) = \tilde{v}_i(n)$$

for $i = 1, 2, 3$. Finally, for any point (x, y) in the subtriangle with vertices $\langle v_1(n), v_2(n), v_3(n) \rangle$, set

$$\delta_n(x, y) = \alpha \tilde{v}_1(n) + \beta \tilde{v}_2(n) + \gamma \tilde{v}_3(n),$$

where

$$(x, y) = \alpha v_1(n) + \beta v_2(n) + \gamma v_3(n).$$

Note that, since the point (x, y) is in the interior of the triangle $\langle v_1(n), v_2(n), v_3(n) \rangle$, we have that

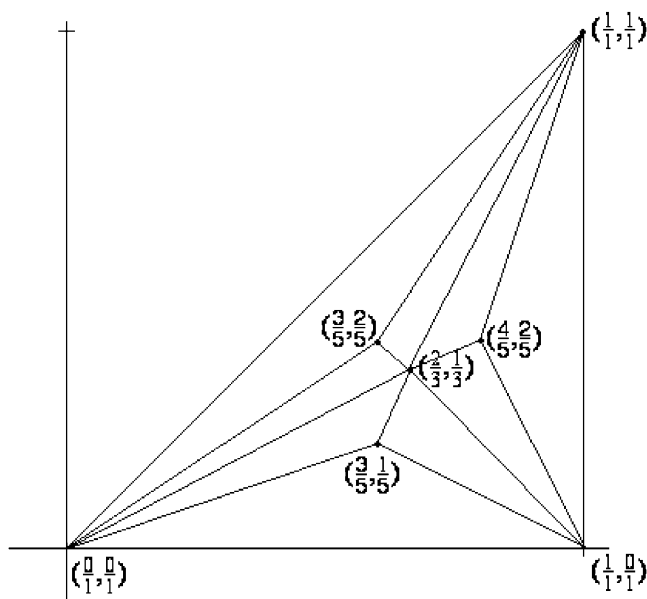
$$\alpha + \beta + \gamma = 1$$

with

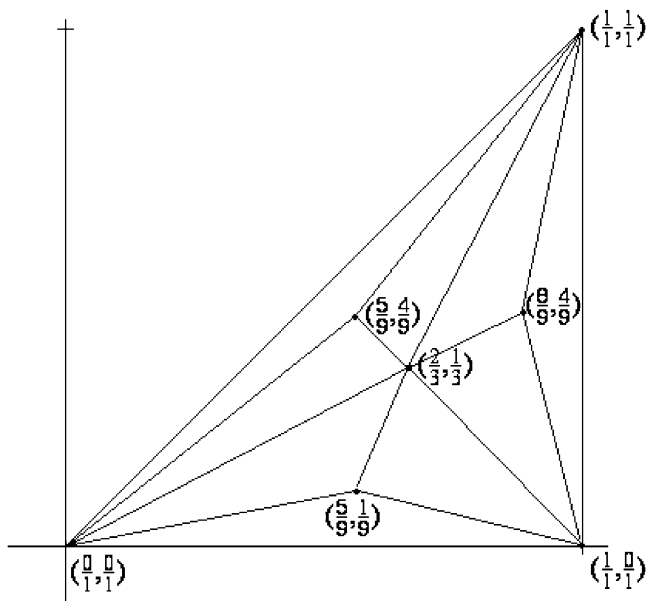
$$0 \leq \alpha, \beta, \gamma \leq 1.$$

As defined, δ_0 and δ_1 are both the identity map since $\Delta_{0,F} = \Delta_{0,B}$ and $\Delta_{1,F} = \Delta_{1,B}$. However, the mappings start to become more complicated with δ_2 .

At this stage we have the Farey partition:



and the Bary partition:



The correspondence between the vertices becomes

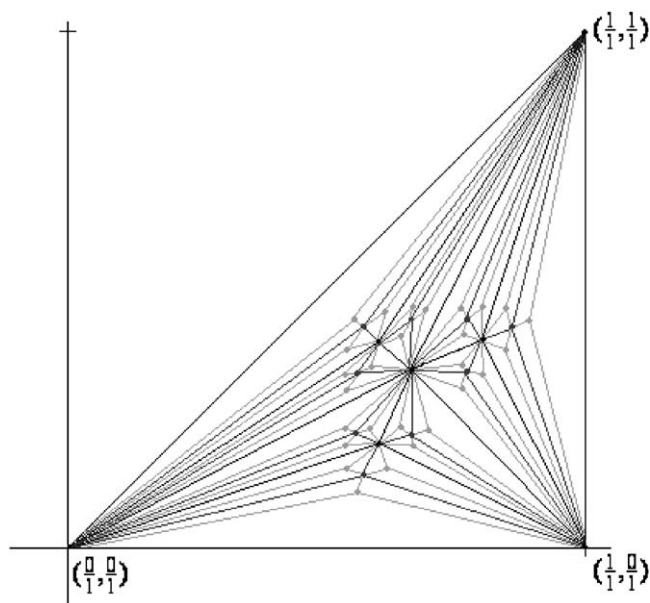
$$\delta_2 \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix},$$

$$\delta_2 \begin{pmatrix} 3/5 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 5/9 \\ 1/9 \end{pmatrix},$$

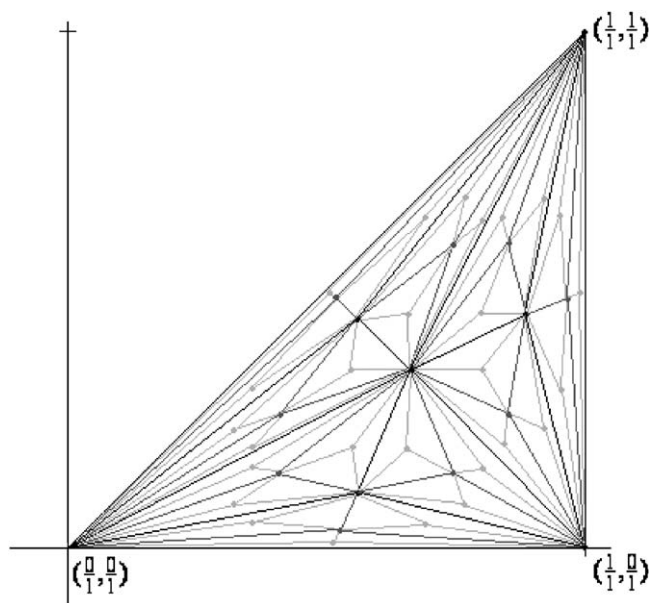
$$\delta_2 \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} = \begin{pmatrix} 8/9 \\ 4/9 \end{pmatrix},$$

$$\delta_2 \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} = \begin{pmatrix} 5/9 \\ 4/9 \end{pmatrix}.$$

Going a few stages further, we get for the Farey partition:



and for the Bary partition:



(We find it interesting that the diagram for the Farey partition is much more aesthetically pleasing than the one for the barycentric partition.)

By definition, $\delta_3(v_i(2)) = \delta_2(v_i(2))$ for any vertex in a subtriangle $\langle v_1(2), v_2(2), v_3(2) \rangle$, of $\Delta_{2,F}$. Thus to describe δ_3 , we need only specify what happens on the new vertices obtained in $\Delta_{3,F}$ and $\Delta_{3,B}$.

This new correspondence becomes:

$$\delta_3\left(\frac{11/27}{4/27}\right) = \left(\frac{5/9}{2/9}\right),$$

$$\delta_3\left(\frac{11/27}{7/27}\right) = \left(\frac{5/9}{3/9}\right),$$

$$\delta_3\left(\frac{14/27}{1/27}\right) = \left(\frac{4/9}{1/9}\right),$$

$$\delta_3\left(\frac{14/27}{13/27}\right) = \left(\frac{4/9}{3/9}\right),$$

$$\delta_3\left(\frac{20/27}{4/27}\right) = \left(\frac{6/9}{2/9}\right),$$

$$\delta_3\left(\frac{20/27}{16/27}\right) = \left(\frac{6/9}{4/9}\right),$$

$$\delta_3\left(\frac{23/27}{7/27}\right) = \left(\frac{7/9}{3/9}\right),$$

$$\delta_3\left(\frac{23/27}{16/27}\right) = \left(\frac{7/9}{4/9}\right),$$

$$\delta_3\left(\frac{26/27}{13/27}\right) = \left(\frac{6/9}{3/9}\right).$$

Definition 4. A point $v \in \Delta_F$ is an interior point of the Farey partitioning if v is not a vertex or on an edge of any triangle of any of the Farey partitions.

Since there are only a countable number of triangles making up the Farey partitioning, we see that almost all points in Δ_F are interior points. It is the interior points that we are interested in. It is these points for which periodicity of our eventual Farey sequences will give candidates for cubic irrationals.

We need to understand a bit better the nature of these interior points before we can define our map δ . Fix an interior point $v \in \Delta_F$. Then for each n , v must be in the

interior of one of the 3^n subtriangles of \mathcal{P}_n . Label this triangle $\Delta_n(v)$. Then we have that, for all m bigger than n ,

$$\Delta_m(v) \subset \Delta_n(v).$$

Thus the interior point v is in a nested sequence of Farey triangles. Further, if $\tilde{\Delta}_n(v)$ denotes the corresponding triangle with respect to the barycentric partitioning, we have another nested sequence of triangles.

Theorem 5. *For each interior point $v \in \Delta_F$, the sequence of $\delta_n(v)$ converges to a point in Δ_B .*

Proof. Fix an interior point $v \in \Delta_F$. For each n , label the vertices of the triangle $\tilde{\Delta}_n(v)$ by $\langle \tilde{v}_1(n), \tilde{v}_2(n), \tilde{v}_3(n) \rangle$. We know, for all $m > n$, that the image $\delta_m(\Delta_m(v)) = \tilde{\Delta}_m(v)$ is contained in $\tilde{\Delta}_n(v)$. Thus the sequence $\delta_n(v)$ will converge if the vertices of the triangles $\delta_n(\tilde{v}_i(n))$ converge.

A calculation shows that the distance from the barycenter of a triangle to any of the vertices is at most two thirds the distance of the longest side of the triangle. If our initial point v is interior, then we can see that at most one of the vertices in $\tilde{\Delta}_n$ will remain constant for infinitely many n . (Usually none remain fixed.) This forces the vertices of the triangles $\tilde{\Delta}_n$ to converge, forcing in turn the sequence $\delta_n(v)$ to converge. \square

Definition 6. Define the Farey–Bary map $\delta: \Delta_F \rightarrow \Delta_B$ by setting

$$\delta(v) = \lim_{n \rightarrow \infty} \delta_n(v)$$

when v is an interior point and

$$\delta(v) = \delta_n(v)$$

when v is a vertex or edge of one of the triangles in $\Delta_{n,F}$, for some n .

While the Farey–Bary map is not continuous at all points, it is so at all interior points, which is almost all points. Again, it is only these interior points that are of interest.

Theorem 7. *Let $\{v_k\}$ be a sequence of points in Δ_F that converge to an interior point v . Then $\{\delta(v_k)\}$ will converge to $\delta(v)$. Thus δ is a continuous function at all interior points.*

Proof. Given any small ball about the point $\delta(v)$, there is an n such that the triangle $\tilde{\Delta}_n(v)$ is contained in this ball. The inverse image of $\tilde{\Delta}_n(v)$ is the triangle $\Delta_n(v)$, of which v is in the interior. Hence, eventually the $\{v_k\}$ must be in $\Delta_n(v)$. Hence their images $\{\delta(v_k)\}$ must eventually be in $\tilde{\Delta}_n(v)$, and hence

must eventually be in our chosen small ball containing $\delta(v)$. Thus $\{\delta(v_k)\}$ converges to $\delta(v)$. \square

4. Farey iteration in the domain as multidimensional continued fraction

Minkowski's $?(x)$ provides a link between algebraic properties of numbers and the failure of differentiability, almost everywhere, for $?(x)$. Our goal is to find analogous links for the Farey–Bary map $\delta(x, y)$. The key algebraic property of the Minkowski question mark function is that $?(x)$ maps quadratic irrationals to rational numbers. The goal of this section is to show that $\delta(x, y)$ maps a class of pairs of cubic irrationals to pairs of rationals. Unfortunately, we cannot make the claim that δ maps all pairs of cubics (even in the same number field) to pairs of rationals.

4.1. Preliminary notation

Let $(\alpha, \beta) \in \Delta_F$ be an interior point. The Farey partitions of Δ_F yields a nested sequence of triangles containing the point (α, β) . (Note that it is not necessarily the case that the vertices of these triangles converge to v .) Suppose that at the n th stage of the Farey partitioning, the triangle that contains (α, β) is $\langle v_1(n), v_2(n), v_3(n) \rangle$. We will maintain the notation $v_i(n)$ to mean either the cartesian version of the vertex or the vector in space that corresponds to the vertex. That is, $v_i(n)$ will refer to $\left(\frac{p_i(n)}{q_i(n)/r_i(n)}\right)$, as well as to $\left(\frac{p_i(n)}{q_i(n)}\right)$. Furthermore, we will order the vertices so that for all n ,

$$r_1(n) \leq r_2(n) \leq r_3(n).$$

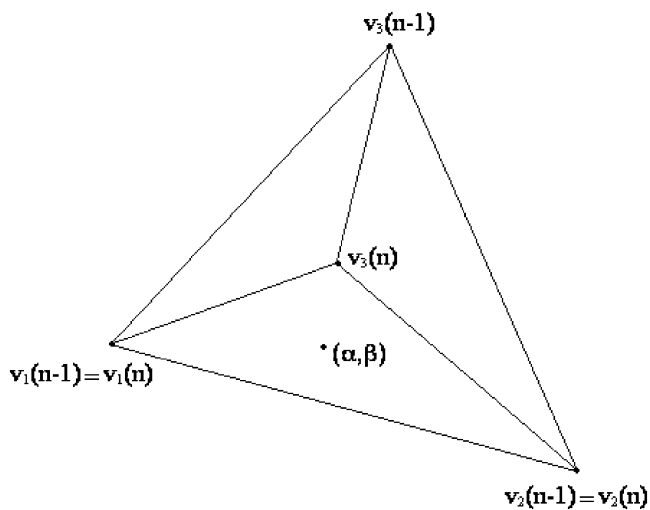
We want to relate the vertices of the $(n-1)$ th subtriangle that contains (α, β) with the vertices of the subtriangle at the next iteration. For that, suppose that $(\alpha, \beta) \in \langle v_1(n-1), v_2(n-1), v_3(n-1) \rangle \subseteq \Delta_{n-1,F}$. Applying the next partition, \mathcal{P}_n , to $\Delta_{n-1,F}$, we decompose $\langle v_1(n-1), v_2(n-1), v_3(n-1) \rangle$ into three new subtriangles. If we let $\langle v_1(n), v_2(n), v_3(n) \rangle$ denote the subtriangle into which (α, β) falls, we see that there are three possibilities for the vertices of $\langle v_1(n), v_2(n), v_3(n) \rangle$.

In case I, the vertices of the newly partitioned triangle will be

$$v_1(n) = v_1(n-1),$$

$$v_2(n) = v_2(n-1),$$

$$v_3(n) = v_1(n-1) + \hat{+} v_2(n-1) \hat{+} v_3(n-1).$$

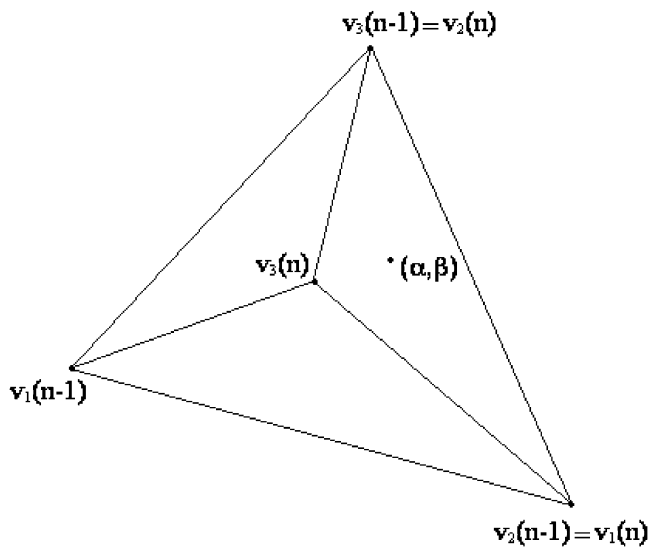


Similarly, the vertices in case II will be

$$v_1(n) = v_2(n-1),$$

$$v_2(n) = v_3(n-1),$$

$$v_3(n) = v_1(n-1) \hat{+} v_2(n-1) \hat{+} v_3(n-1).$$

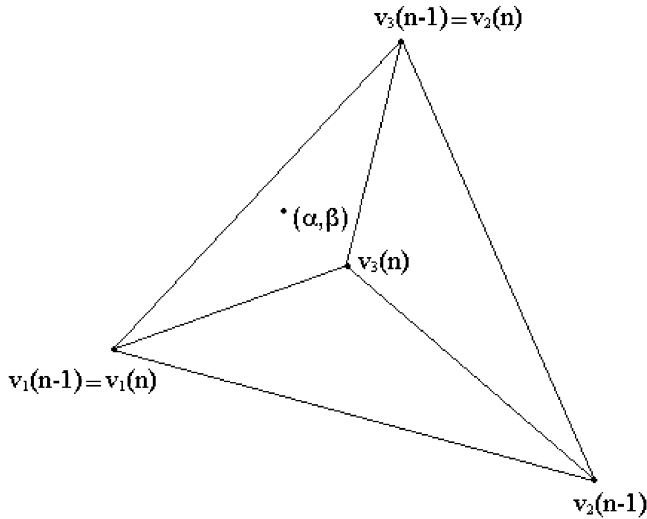


For case III we have

$$v_1(n) = v_1(n-1),$$

$$v_2(n) = v_3(n-1),$$

$$v_3(n) = v_1(n-1) \hat{+} v_2(n-1) \hat{+} v_3(n-1).$$



In the next section, we will streamline this notation.

4.2. Fixing notation

For each interior point (α, β) in Δ_F we now associate a sequence of positive integers that will uniquely determine the nested sequence of Farey subtriangles containing (α, β) .

To motivate the eventual notation, consider the following three possibilities. Start with a triangle, with vertices v_1 , v_2 , and v_3 , still keeping the convention that $r_1 \leq r_2 \leq r_3$. Suppose we perform k type I operations in a row. The resulting new triangle will have vertices in the following form:

$$v_1, v_2, kv_1 \hat{+} kv_2 \hat{+} v_3.$$

If we perform a type II operation on the triangle, and then $k-1$ type I operations, the new triangle will have vertices:

$$v_2, v_3, v_1 \hat{+} kv_2 \hat{+} kv_3.$$

If we perform a type III operation on the triangle, and then $k - 1$ type I operations, the new triangle will have vertices:

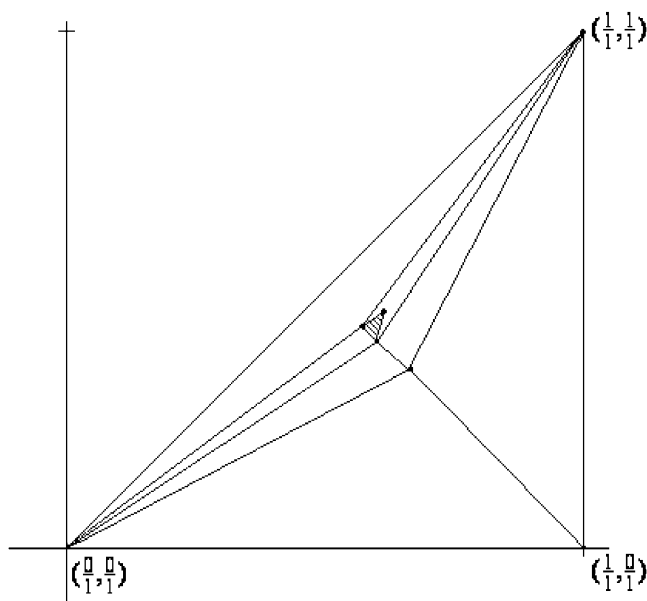
$$v_1, v_3, kv_1 + v_2 + kv_3.$$

This suggests the following notation.

Define a sequence $\{a_1(i_1), a_2(i_2), \dots\}$ to be such that each $a_k(i_k)$ is a positive integer and each i_k represents either cases I, II or III. The value of $a_k(i_k)$ denotes the operation of first applying a type i_k operation and then $a_k(i_k) - 1$ type I operations. We use the further convention that for $k \geq 2$, i_k can only be of type II or III.

Note that, in the notation of the previous section, by the time we are at step $a_k(i_k)$, we have performed $n = a_1(i_1) + a_2(i_2) + \dots + a_k(i_k)$ Farey partitions of \triangle_F . We associate to each interior point $(\alpha, \beta) \in \triangle_F$ the sequence that yields the corresponding Farey partitions that converge to (α, β) . This sequence will be unique. We will also use $\langle v_1(k), v_2(k), v_3(k) \rangle$ to denote the subtriangle of $\triangle_{n,F}$ containing (α, β) after $n = a_1(i_1) + a_2(i_2) + \dots + a_k(i_k)$ steps partitioning \triangle_F . Finally, if we know what case we are in, that is, if we know i_k , we will simply write a_k instead of $a(i_k)$.

Example 8. The shaded region below corresponds to all points $(2(III), 1(II), 1(I))$. Note that in the notation of last section $n = 4$ but that in the notation of this section and for the rest of the paper $k = 3$.



We now have the following recursion formulas for the vertices. For case I at the k th step we get

$$v_1(k) = v_1(k-1),$$

$$v_2(k) = v_2(k-1),$$

$$v_3(k) = a_k v_1(k-1) \hat{+} a_k v_2(k-1) \hat{+} v_3(k-1),$$

For case II we have

$$v_1(k) = v_2(k-1),$$

$$v_2(k) = v_3(k-1),$$

$$v_3(k) = v_1(k-1) \hat{+} a_k v_2(k-1) \hat{+} a_k v_3(k-1).$$

Finally, for case III we get

$$v_1(k) = v_1(k-1),$$

$$v_2(k) = v_3(k-1),$$

$$v_3(k) = a_k v_1(k-1) \hat{+} v_2(k-1) \hat{+} a_k v_3(k-1).$$

We can put these recursion relations naturally into a matrix language. At each step k , define M_k to be the three-by-three matrix

$$M_k = \begin{pmatrix} v_1(k) & v_2(k) & v_3(k) \end{pmatrix} = \begin{pmatrix} p_1(k) & p_2(k) & p_3(k) \\ q_1(k) & q_2(k) & q_3(k) \\ r_1(k) & r_2(k) & r_3(k) \end{pmatrix}.$$

If, from the $(k-1)$ th step to the k th step, we are in case I, then

$$M_k = M_{k-1} \begin{pmatrix} 1 & 0 & a_k \\ 0 & 1 & a_k \\ 0 & 0 & 1 \end{pmatrix},$$

for case II we have

$$M_k = M_{k-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & a_k \\ 0 & 1 & a_k \end{pmatrix},$$

and for case III,

$$M_k = M_{k-1} \begin{pmatrix} 1 & 0 & a_k \\ 0 & 0 & 1 \\ 0 & 1 & a_k \end{pmatrix}.$$

Denote in each of these cases the matrix on the right by $A_k(I)$, $A_k(II)$ and $A_k(III)$, respectively. Then we have that each M_k is the product of M_0 with a sequence of various A_m .

Theorem 9. *Each M_k is in the special linear group $\mathbf{SL}(3, \mathbf{Z})$.*

Proof. All we need to show is that for all k ,

$$\det(M_k) = \pm 1.$$

This follows immediately from observing that $\det(M_0) = 1$ and that the determinants of each of the various $A_k(I)$, $A_k(II)$ and $A_k(III)$ are also plus or minus one. \square

4.3. Areas of Farey subtriangles

Given a finite sequence $\{a_1(i_1), a_2(i_2), \dots, a_k(i_k)\}$ of positive integers, we define

$$\Delta_k = \{(x, y) : \{a_1(i_1), \dots, a_k(i_k)\} \text{ are the 1st } k \text{ terms in Farey sequence}\}.$$

A major goal of this paper is showing that the areas of these triangles Δ_k cannot go to zero too quickly. For these calculations, we will need an easy formula for the areas of the Δ_k .

Theorem 10. *The area of a triangle with vertices $(p_1/r_1, q_1/r_1)$, $(p_2/r_2, q_2/r_2)$, and $(p_3/r_3, q_3/r_3)$ is*

$$\text{Area of triangle} = \frac{1}{2} \frac{\left| \det \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix} \right|}{r_1 r_2 r_3}.$$

This is just a calculation involving cross products.

Corollary 11. *Given any finite sequence $\{a_1(i_1), a_2(i_2), \dots, a_k(i_k)\}$ of positive integers,*

$$2\text{Area}(\Delta_k) = \frac{1}{r_1(k)r_2(k)r_3(k)}.$$

This follows since $\det(M_k) = \pm 1$.

4.4. Farey periodicity implies cubic irrationals

We want to determine when our Farey sequence can be used to determine algebraic properties of the corresponding interior point (α, β) . We want to show:

Theorem 12. *Suppose that the interior point $(\alpha, \beta) \in \Delta_F$ has an eventually periodic Farey sequence. Further suppose that its nested sequence of Farey triangles converges to this point (α, β) . Then both α and β are algebraic numbers with $\deg(\alpha) \leq 3$, $\deg(\beta) \leq 3$ and*

$$\dim_{\mathbf{Q}} \mathbf{Q}[\alpha, \beta] \leq 3.$$

This is why the Farey partitioning can be viewed as a multi-dimensional continued fraction algorithm.

Proof. We will be heavily using two facts. First, an eigenvector $(1, a, b)$ of a 3×3 matrix with rational coefficients has the property that

$$\dim_{\mathbf{Q}} \mathbf{Q}[a, b] \leq 3,$$

as seen in a similar argument in [8] in Section 8. Second, if we multiply a matrix, which has a largest real eigenvalue, repeatedly by itself, in the limit the columns of the matrix converge to the eigenvector corresponding to the largest eigenvalue, unless one of the columns is itself an eigenvector for another eigenvalue.

Suppose that $(\alpha, \beta) \in \Delta$ has an eventually periodic Farey sequence. By assumption, the vertices of the corresponding Farey partition triangles converge to the point (α, β) . We have seen above that the vertices of the partition triangles correspond to the columns of matrices that are the products of various $A_k(I)$, $A_k(II)$ and $A_k(III)$. With the assumption of periodicity, denote the product of the initial non-periodic matrices be B and the product of the periodic part be A . Then some of the Farey partition triangles about the point (α, β) are given by

$$B, BA, BA^2, BA^3, \dots$$

The columns of the matrices A, A^2, A^3, \dots must converge to a multiple of $B^{-1}(1, \alpha, \beta)^T$. But the columns of the A^k must also converge to an eigenvector and hence $B^{-1}(1, \alpha, \beta)^T$ is an eigenvector of the matrix A . This will give us that α and β must have the desired properties. \square

Note that the assumption that the vertices of the Farey partition triangles converging to (α, β) is needed. For example, if the Farey sequence to (α, β) eventually consists solely of terms $a_k(III)$, then the vertices will converge to a line segment, not a point. This follows from the corresponding matrices. If beyond a certain point, we just have terms $a_k(III)$, then we are eventually

multiplying together matrices of the form

$$\begin{pmatrix} 1 & 0 & a_k \\ 0 & 0 & 1 \\ 0 & 1 & a_k \end{pmatrix}.$$

The presence of the first column being a one, then two zeros, means that beyond a certain point, all of the nested Farey partition triangles contain the same vertex. The other two vertices will of course converge to a point. On the other hand, it can be checked that this is the only way for the Farey partition triangles to not converge to a single point.

5. Iteration in the barycentric range

We have defined Farey partitions, \mathcal{P}_n , in Δ_F and barycentric partitions, $\tilde{\mathcal{P}}_n$ in Δ_B . In Δ_F , the Farey partitions yielded an association between each interior point (α, β) and a sequence obtained from the convergence of the subtriangles resulting from successive applications of the partitions, \mathcal{P}_n . This association depended only on the successive partitioning of each subtriangle into three more subtriangles and not on the relative positioning of the new subtriangles. We can follow the same procedure in Δ_B . That is, if we let $(a, b) \in \Delta_B$ be the image of an interior point, we can again associate with (a, b) a sequence of positive integers $\{\tilde{a}_1(i_1), \tilde{a}_2(i_2), \dots\}$ which come from a sequence of barycentric triangles converging to the point (a, b) .

Label the triangle corresponding to $\{\tilde{a}_1(i_1), \tilde{a}_2(i_2), \dots, \tilde{a}_k(i_k)\}$ by

$$\tilde{\Delta} \{\tilde{a}_1(i_1), \tilde{a}_2(i_2), \dots, \tilde{a}_k(i_k)\}.$$

Recall that

$$\tilde{\Delta}_B = \langle \tilde{v}_1(0), \tilde{v}_2(0), \tilde{v}_3(0) \rangle,$$

where

$$\tilde{v}_1(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \tilde{v}_2(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \tilde{v}_3(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Associated with the sequence $\{\tilde{a}_1(i_1), \tilde{a}_2(i_2), \dots, \tilde{a}_k(i_k)\}$ will be vertices $v_1(k), v_2(k)$ and $v_3(k)$ and corresponding vectors

$$v_1(k) = \begin{pmatrix} * \\ * \\ 3^{a_1+\dots+a_k} \end{pmatrix}, \quad v_2(k) = \begin{pmatrix} * \\ * \\ 3^{a_1+\dots+a_k} \end{pmatrix}, \quad v_3(k) = \begin{pmatrix} * \\ * \\ 3^{a_1+\dots+a_k} \end{pmatrix},$$

where the other entries for the vectors are nonnegative integers. There are, of course, matrices \tilde{M}_n that map the vertices from a given level to the vertices of the next level, in analogue to the matrices M_n . The \tilde{M}_n are products of matrices of the form

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

depending if we are in cases I, II or III, respectively.

Note that at each individual step of the barycentric partitioning, we are cutting the area down by a factor of a $1/3$. This leads to the following theorem.

Theorem 13. *Twice the area of $\tilde{\Delta} \{\tilde{a}_1(i_1), \tilde{a}_2(i_2), \dots, \tilde{a}_k(i_k)\}$ is $\frac{1}{3^{a_1+a_2+\dots+a_k(i_k)}}$.*

5.1. Ternary periodicity implies rationality

Suppose that we have a point $(a, b) \in \Delta_B$ for which the barycentric partitioning is eventually periodic. We want to show that both a and b are rational numbers. That is, we want the following theorem.

Theorem 14. *Let $(a, b) \in \Delta_B$ be the image of an interior point. If (a, b) has an eventually periodic Barycentric sequence, then both a and b are rational.*

Proof. This proof is almost exactly the same as the corresponding proof for the Farey case, whose notation we adopt. There is one significant difference, namely that the matrices whose columns yield the vertices of the barycentric partitioning are all multiples of a stochastic matrix. This means that each matrix is a multiple of a matrix whose columns add to one. If the columns add to one, then it can easily be shown that the limit of the products of such a matrix converges to a matrix whose rows are multiples of $(1, 1, 1)$ (see Chapter 6 in [26]). Thus the matrices A, A^2, A^3, \dots converge to a matrix whose rows are multiples of $(1, 1, 1)$. Since everything in sight is rational, we can show that $B^{-1}(1, \alpha, \beta)^T$ will converge to a triple of rational numbers. Since the entries of B are integers, this yields that α and β are rational numbers. \square

6. The Farey–Bary analog of singularness

The original Minkowski $\eta(x)$ function is singular, meaning that even though it is increasing and continuous, it has derivative zero almost everywhere. The key to the proof lies in showing that at almost all points

$$\liminf_{k \rightarrow \infty} \frac{\text{length of interval in range}}{\text{length of interval in domain}} = 0,$$

for appropriately defined intervals. We will show a direct analog of this, where the lengths of intervals are replaced by areas of triangles. Thus we will show

$$\liminf_{k \rightarrow \infty} \frac{\text{area of subtriangle in range}}{\text{area of subtriangle in domain}} = 0,$$

again for appropriately defined subtriangles.

6.1. *Almost everywhere* $\limsup \frac{a_1 + \dots + a_n}{n} = \infty$

This is the most technically difficult section of the paper. The goal is to show the following theorem, which will be critical in the next section. Recall that given any interior point $(\alpha, \beta) \in \Delta$, we have associated a sequence $\{a_1, a_2, \dots\}$ of positive integers. (Also, we know that the noninterior points in Δ have measure zero; for this reason we will ignore these noninterior points.) We want to show that this sequence of positive integers must increase to infinity, in some sense, almost everywhere. The precise statement is:

Theorem 15. *The set of $(\alpha, \beta) \in \Delta$ for which*

$$\limsup_{n \rightarrow \infty} \frac{a_1 + \dots + a_n}{n} < \infty$$

has measure zero.

As it will only be apparent in the next section why to we need this theorem, we recommend on the first reading of this paper to go to the next section first.

Before proving the theorem, we need a preliminary lemma. First, let

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

and let

$$T = \langle v_1, v_2, v_3 \rangle$$

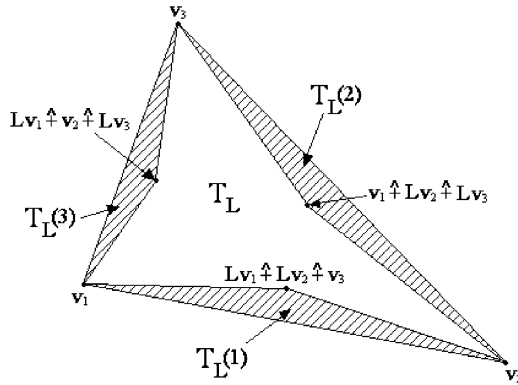
be the corresponding triangle in the plane, where the v_i are now viewed as points in the plane.

Suppose that $\det v_1 v_2 v_3 = 1$. Then we know that

$$2\text{area of } \langle v_1, v_2, v_3 \rangle = \frac{1}{z_1 z_2 z_3}.$$

Given a positive number $L > 1$, define $T_L(1)$ to be the triangle with vertices \hat{v}_1, \hat{v}_2 and $Lv_1 + \hat{Lv}_2 + \hat{v}_3$, $T_L(2)$ the triangle with vertices $\hat{v}_2, v_1 + \hat{Lv}_2 + \hat{Lv}_3$ and \hat{v}_3 and $T_L(3)$

the triangle with vertices $\hat{v}_1, Lv_1 + v_2 + Lv_3$ and \hat{v}_3 .



Define

$$T_L = T - T_L(1) - T_L(2) - T_L(3).$$

We now state and then prove a lemma that is the technical heart of the proof of the theorem:

Lemma 16. *For all $L \geq 1$,*

$$\text{area}(T - T_L) \leq \frac{L-1}{L} \text{area}(T).$$

Proof. We know that

$$2\text{area}(T) = \frac{1}{z_1 z_2 z_3}.$$

For ease of notation, we set $z_1 = x$, $z_2 = y$, $z_3 = z$. Then

$$\begin{aligned} 2\text{area}(T - T_L) &= \frac{1}{xyz} - \frac{1}{x(Lx + y + Lz)z} - \frac{1}{xy(Lx + Ly + z)} - \frac{1}{(x + Ly + Lz)y} \\ &= \frac{1}{xyz} \left[1 - \frac{y}{(Lx + y + Lz)} - \frac{z}{(Lx + Ly + z)} - \frac{x}{(x + Ly + Lz)} \right]. \end{aligned}$$

Thus we must show that

$$\left[1 - \frac{y}{(Lx + y + Lz)} - \frac{z}{(Lx + Ly + z)} - \frac{x}{(x + Ly + Lz)} \right] \leq \frac{L-1}{L}.$$

Setting

$$\alpha = x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2,$$

we have that

$$\begin{aligned} & \left[1 - \frac{y}{(Lx + y + Lz)} - \frac{z}{(Lx + Ly + z)} - \frac{x}{(x + Ly + Lz)} \right] \\ &= \frac{\alpha(L^3 - L) + 2xyz(L^3 - 1)}{(Lx + Ly + z)(Lx + y + Lz)(x + Ly + Lz)}. \end{aligned}$$

After a series of calculations, we get that this is equal to

$$(L - 1) \left[\frac{L(L + 1)\alpha + 2xyz(L^2 + L + 1)}{L(L^2 + L + 1)\alpha + L^2(x^3 + y^3 + z^3) + xyz(3L^2 + 2L^3 + 1)} \right].$$

Thus we must show that

$$\left[\frac{L(L + 1)\alpha + 2xyz(L^2 + L + 1)}{L(L^2 + L + 1)\alpha + L^2(x^3 + y^3 + z^3) + xyz(3L^2 + 2L^3 + 1)} \right] \leq \frac{1}{L},$$

which is equivalent to showing that

$$\begin{aligned} & L^2(L + 1)\alpha + 2xyzL(L^2 + L + 1) \\ & \leq L(L^2 + L + 1)\alpha + L^2(x^3 + y^3 + z^3) + xyz(2L^3 + 3L^2 + 1), \end{aligned}$$

which in turn, reduces to showing that

$$2Lxyz \leq \alpha + L^2(x^3 + y^3 + z^3) + L^2xyz + xyz.$$

This last inequality follows from the fact that $L \geq 1$. Thus the proof of the lemma is done.

Proof of Theorem. For each positive integer N , set

$$M_N = \left\{ (\alpha, \beta) \in \Delta : \text{for all } n \geq 1, \frac{a_1 + \cdots + a_n}{n} \leq N \right\}.$$

We will show that

$$\text{measure}(M_N) = 0.$$

Since the union of all of the M_N is the set we want to show has measure zero, we will be done.

Now, $\frac{a_1 + \cdots + a_n}{n} \leq N$ if and only if

$$a_1 + \cdots + a_n \leq nN.$$

Since each $a_i \geq 1$, this last inequality implies

$$n - 1 + a_n \leq nN$$

or

$$a_n \leq n(N - 1) + 1.$$

Set

$$\tilde{M}_N = \{(\alpha, \beta) \in \Delta : \text{for all } n \geq 1, a_n \leq n(N - 1) + 1\}.$$

Since $M_N \subset \tilde{M}_N$, if we can show that $\text{measure}(\tilde{M}_N) = 0$, we will be done.

Set

$$\tilde{M}_N(1) = \{(\alpha, \beta) \in \Delta : a_1 \leq (N - 1) + 1\}$$

and in general

$$\tilde{M}_N(k) = \{(\alpha, \beta) \in \tilde{M}_N(k - 1) : a_k \leq k(N - 1) + 1\}.$$

Then we have a decreasing nested sequence of sets with

$$\tilde{M}_N = \bigcap_{k=1}^{\infty} \tilde{M}_N(k).$$

But this puts us into the language of the above lemma. Letting $L = k(N - 1) + 1$, we can conclude that

$$\text{measure}(\tilde{M}_N(k)) = \frac{k(N - 1)}{k(N - 1) + 1} \text{measure}(\tilde{M}_N(k - 1))$$

and hence

$$\text{measure}(\tilde{M}_N) = \prod_{k=2}^{\infty} \frac{k(N - 1)}{k(N - 1) + 1}.$$

We must show this infinite product is zero, which is equivalent to showing that its reciprocal

$$\prod_{k=2}^{\infty} \frac{k(N - 1) + 1}{k(N - 1)} = \prod_{k=2}^{\infty} \left(1 + \frac{1}{k(N - 1)}\right) = \infty.$$

Taking logarithms, this is the same as showing that the series

$$\sum_{k=2}^{\infty} \log\left(1 + \frac{1}{k(N - 1)}\right) = \infty.$$

This in turn follows since, for large enough k , we have

$$\log\left(1 + \frac{1}{k(N-1)}\right) \geq \frac{1}{2k(N-1)}.$$

We are done.

6.2. Almost everywhere

$$\liminf(\text{area}(\tilde{\Delta}_n)/\text{area}(\Delta_n)) = 0.$$

The goal of this section, and for the entire paper, is:

Theorem 17. *For any point $(\alpha, \beta) \in \Delta$, off of a set of measure zero,*

$$\liminf_{n \rightarrow \infty} \frac{\text{area}(\tilde{\Delta}\{a_1(i_1), a_2(i_2), \dots, a_n(i_n)\})}{\text{area}(\Delta\{a_1(i_1), a_2(i_2), \dots, a_n(i_n)\})} = 0.$$

This is capturing the intuition that the determinant of the Jacobian of the map $\delta: \Delta_F \rightarrow \Delta_B$ is zero almost everywhere, which in turn is a direct generalization that the Minkowski question-mark function is singular. In fact, our proof is in spirit a generalization of Viader, Paradis and Bibiloni's work in [42].

Proof. We know that, letting $s_n = a_1 + \dots + a_n$,

$$\text{area}(\tilde{\Delta}\{a_1(i_1), a_2(i_2), \dots, a_n(i_n)\}) = \frac{1}{2 \cdot 3^{s_n}}$$

and that

$$\text{area}(\Delta\{a_1(i_1), a_2(i_2), \dots, a_n(i_n)\}) = \frac{1}{2 \cdot r_1(n)r_2(n)r_3(n)}.$$

Thus we want to show that, almost everywhere,

$$\liminf_{n \rightarrow \infty} \frac{r_1(n)r_2(n)r_3(n)}{3^{s_n}} = 0.$$

We know that $r_3(n)$ is

$$a_n r_1(n-1) + a_n r_2(n-1) + r_3(n-1),$$

$$a_n r_1(n-1) + r_2(n-1) + a_n r_3(n-1)$$

or

$$r_1(n-1) + a_n r_2(n-1) + a_n r_3(n-1).$$

Thus we have, by the convention of our notation,

$$r_1(n) \leq r_2(n) \leq r_3(n) \leq (2a_n + 1)r_3(n-1).$$

By iterating this inequality, we have

$$r_1(n) \leq r_2(n) \leq r_3(n) \leq \prod_{i=1}^n (2a_i + 1).$$

Thus

$$\frac{\text{area}(\tilde{\Delta}\{a_1(i_1), a_2(i_2), \dots, a_n(i_n)\})}{\text{area}(\Delta\{a_1(i_1), a_2(i_2), \dots, a_n(i_n)\})} \leq \frac{\prod_{i=1}^n (2a_i + 1)^3}{3^{s_n}}.$$

By the arithmetic-geometric mean,

$$\prod_{i=1}^n (1 + b_i) \leq \left(1 + \frac{b_1 + \dots + b_n}{n}\right)^n.$$

Setting $b_j = 2a_j$, we get

$$\begin{aligned} \frac{\text{area}(\tilde{\Delta}\{a_1(i_1), a_2(i_2), \dots, a_n(i_n)\})}{\text{area}(\Delta\{a_1(i_1), a_2(i_2), \dots, a_n(i_n)\})} &\leq \frac{(1 + \frac{2s_n}{n})^{3n}}{3^{s_n}} \\ &\leq \frac{(\frac{3s_n}{n})^{3n}}{3^{s_n}} \\ &\leq \left(\frac{27 \cdot (\frac{s_n}{n})^3}{3^{s_n/n}}\right)^n. \end{aligned}$$

From the previous section, we know that $s_n/n \rightarrow \infty$, almost everywhere. Since the above denominator has a $3^{s_n/n}$ term while the numerator only has a $(s_n/n)^3$ term, the entire ratio must approach zero, giving us our result.

7. Questions

There are a number of natural questions. First, all of this can almost certainly be generalized to higher dimensions.

More importantly, how much does the function theory of δ influence the diophantine properties of points in Δ ?

There are many multi-dimensional continued fraction algorithms. For any of these that involve partitioning a given triangle into three new subtriangles, a map analogous to our δ can of course be defined. What are the properties of these new maps?

Underlying most work on multidimensional continued fractions, though frequently hidden behind view, are Lie theoretic properties of the special linear group. Can this be made more explicit?

Finally, the initial Hermite problem remains open.

Acknowledgments

We thank Lori Pedersen for making all of the diagrams. We thank Andrew Marder for pointing out an error on an earlier version of this paper and would like to thank Keith Briggs for pointing out some errors in the bibliography in another earlier version.

References

- [1] S. Assaf, L. Chen, T. Cheslack-Postava, B. Cooper, A. Diesl, T. Garrity, M. Lepinski, A. Schuyler, A dual approach to triangle sequences: a multidimensional continued fraction algorithm, preprint, available at: <http://xxx.lanl.gov/abs/math.NT/0206105>.
- [2] L. Bernstein, The Jacobi–Perron Algorithm-its Theory and Application, Lecture Notes in Mathematics, Vol. 207, Springer, Berlin, 1971.
- [3] A.J. Brentjes, Multi-dimensional Continued Fraction Algorithms, Mathematical Centre Tracts 145, Mathematisch Centrum, Amsterdam, 1981.
- [4] A. Denjoy, Sur une fonction reelle de Minkowski, *J. Math. Pures Appl.* 17 (1938) 105–151.
- [5] A. Denjoy, Sur une fonction de Minkowski, *C. R. Acad. Sci. Paris* 194 (1932) 44–46.
- [6] S. Ferenczi, C. Holton, L. Zamboni, Structure of three interval exchange transformations I: an arithmetic study, 1999, *Ann. Inst. Fourier (Grenoble)* 51 (4) (2001) 861–901.
- [7] H.R.P. Ferguson, R.W. Forcade, Generalization of the Euclidean algorithm for real numbers to all dimensions higher than two, *Bull. Amer. Math. Soc.* 1 (6) (1979) 912–914.
- [8] T. Garrity, On periodic sequences for algebraic numbers, *J. Number Theory* 88 (1) (2001) 83–103.
- [9] R. Girgensohn, Constructing singular functions via Farey fractions, *J. Math. Anal. Appl.* 203 (1996) 127–141.
- [10] D. Grabiner, Farey nets and multidimensional continued fractions, *Monatsh. Math.* 114 (1n) (1992) 35–61.
- [11] D. Grabiner, J. Lagarias, Cutting sequences for geodesic flow on the modular surface and continued fractions, <http://arxiv.org/abs/math.DS/9707215>.
- [12] A. Gupta and A. Mittal, Bifurcating continued fractions, 2000, <http://front.math.ucdavis.edu/math.GM/0002227>.
- [13] R. Güting, Zur Verallgemeinerung des Kettenbruchalgorithmus. I, *J. Reine Angew. Math.* 278/279 (1975) 165–173.
- [14] C. Hermite, Letter to C.D.J. Jacobi, *J. Reine Angew. Math.* 40 (1839) 286.
- [15] H. Hancock, Development of the Minkowski Geometry of Numbers, Vols. I and II, Dover, New York, 1964.
- [16] J.R. Kinney, Note on a singular function of Minkowski, *Proc. Amer. Math. Soc.* 11 (5) (1960) 788–794.
- [17] F. Klein, Über eine geometrische Auffassung der gewöhnlichen Kettenbruchentwicklung, *Gesammelte Mathematische Abhandlungen*, Vol. 2, Springer, 1922, pp. 209–213.
- [18] L. Kollros, Un Algorithmes pour L’Aproximation simultanée de Deux Grandueurs, Inaugural-Dissertation, Universität Zürich, 1905.

- [19] E. Korkina, La périodicité des fractions continues multidimensionnelles, C. R. Acad. Sci. Paris Sr. I Math. 319 (8) (1994) 777–780.
- [20] E. Korkina, The simplest 2-dimensional continued fraction, J. Math. Sci. 82 (5) (1996) 3680–3685.
- [21] E. Korkina, Two-dimensional continued fractions. The simplest examples, Trudy Mat. Inst. Steklov. 209 (1995) (in Russian) (Osob. Gladkikh Otobrazh. s Dop. Strukt., 143–166).
- [22] G. Lachaud, Klein Polygons and Geometric Diagrams, Contemporary Mathematics, Vol. 210, Amer. Math. Soc., Providence, RI, 1998, pp. 365–372.
- [23] G. Lachaud, Sails and Klein Polyhedra, Contemporary Mathematics, Vol. 210, Amer. Math. Soc., Providence, RI, 1998, pp. 373–385.
- [24] J. Lagarias, Geodesic multidimensional continued fractions, Proc. London Math. Soc. 69 (1994) 464–488.
- [25] K. Mahler, Inequalities for ideal bases in algebraic number fields, J. Australian Math. Soc. 4 (1964) 425–448.
- [26] H. Minc, Nonnegative Matrices, Wiley, New York, 1988.
- [27] H. Minkowski, Gesammelte Abhandlungen, AMS-Chelsea, New York, 1967.
- [28] H. Minkowski, Ein Kriterium für algebraischen Zahlen, *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse*, pp. 64–88. (Also in Gesammelte Abhandlungen, Chelsea, pp. 293–315.)
- [29] H. Minkowski, Über periodische Approximationen algebraischer Zahlen, Acta Math. 26, 333–351. (Also in Gesammelte Abhandlungen, Chelsea, pp. 357–371.)
- [30] Von R. Monkemeyer, Über Fareynetze in n Dimensionen, Math. Nach. 11 (1954) 321–344.
- [31] J. Paradis, P. Viader, L. Bibiloni, The derivative of Minkowski's $\varphi(x)$ function, J. Math. Anal. Appl. 253 (2001) 107–125.
- [32] H. Poincaré, Sur une Generalisation des Fractions Continues, Oeuvres de Henri Poincare, tome V, Gauthier-Villar, Paris, pp. 185–188.
- [33] G. Ramharter, On Minkowski's singular function, Proc. Amer. Math. Soc. 99 (3) (1987) 596–597.
- [34] F. Ryde, Arithmetical continued fractions, Lunds Univ. Arsskrift N. F. Avd. 2 22 (2) (1926) 1–182.
- [35] F. Ryde, On the relation between two Minkowski functions, J. Number Theory 77 (1983) 47–51.
- [36] R. Salem, On some singular monotonic functions which are strictly increasing, Trans. Amer. Math. Soc. 53 (1943) 427–439.
- [37] F. Schweiger, The Metrical Theory of Jacobi–Perron Algorithm, Lecture Notes in Mathematics, Vol. 334, Springer, Berlin, 1973.
- [38] F. Schweiger, Über einen Algorithmus von R. Güting, J. Reine Angew. Math. 293/294 (1977) 263–270.
- [39] F. Schweiger, Ergodic Theory of Fibred Systems and Metric Number Theory, Oxford University Press, Oxford, 1995.
- [40] F. Schweiger, Multidimensional Continued Fractions, Oxford University Press, Oxford, 2000.
- [41] R. Tichy, J. Uitz, An extension of Minkowski's singular function, Appl. Math. Lett. 8 (5) (1995) 139–46.
- [42] P. Viader, J. Paradis, L. Bibiloni, A new light on Minkowski's $\varphi(x)$ function, J. Number Theory 73 (1998) 212–227.